Parabolic nondiffracting optical wave fields

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We demonstrate the existence of parabolic beams that constitute the last member of the family of fundamental nondiffracting wave fields and determine their associated angular spectrum. Their transverse structure is described by parabolic cylinder functions, and contrary to Bessel or Mathieu beams their eigenvalue spectrum is continuous. Any nondiffracting beam can be constructed as a superposition of parabolic beams, since they form a complete orthogonal set of solutions of the Helmholtz equation. A novel class of traveling parabolic waves is also introduced for the first time. © 2004 Optical Society of America

The spatial evolution of propagation-invariant optical fields (PIOFs) has been a subject of interest, since under ideal conditions they propagate indefinitely without change of their transverse intensity distribution. PIOFs have an extensive range of applications in fields such as optical tweezers, metrology, microlithography, and wireless and optical communications, among others.1,2

The three-dimensional Helmholtz equation (HE) is known to be separable in 11 orthogonal coordinate systems.3 Of them, only the rectangular (i.e., Cartesian), circular cylindrical, elliptic cylindrical, and parabolic cylindrical coordinates have translation symmetries and allow the HE to be separated into a transverse part and a longitudinal part.3–5 This separability of the equation imposes the condition that the solutions of the transverse part are independent of the longitudinal coordinate. PIOFs are exact solutions of the HE and have been demonstrated theoretically and experimentally for the first three coordinate systems: They are plane waves in Cartesian coordinates, Bessel beams in circular cylindrical coordinates,1 and Mathieu beams in elliptic coordinates.6–8 However, the fundamental family of PIOFs in parabolic coordinates had remained unexplored until now.

We present the last class of PIOFs that are exact solutions of the HE. These solutions are expressed in terms of parabolic cylinder functions and have an inherent geometry imposed by the parabolic coordinates. By use of the group symmetries of the HE, a spectrum for these fields is found that, common to all PIOFs, lies on a ring.9 The solutions describe satisfactorily transverse parabolic stationary waves, and with them a novel class of traveling waves is constructed.

Fundamental PIOFs of the HE can be expressed in terms of a reduced Whittaker integral10

\[ U(r) = \exp(-ik_z z) \int_{-\pi}^{\pi} A(\varphi) \times \exp(-ik_z (x \cos \varphi + y \sin \varphi)) d\varphi, \]  

where \( A(\varphi) \) is the angular spectrum of field \( U(r) \) defined on a ring of radius \( k_z \) in frequency space. The transverse and longitudinal wave-vector components satisfy the relation \( k_z = k_x^2 + k_y^2 \), where \( k \) is the wave number. We assume a time dependence \( \exp(i\omega t) \) for the optical fields. From Eq. (1) it is clear that the field intensity \( I \sim |U|^2 \) is independent of propagation coordinate \( z \).

The known fundamental solutions of the HE that describe PIOFs are fully determined after their angular spectrum is introduced in Eq. (1). For \( m \) plane waves we have \( A(\varphi) = \sum_{\pm} \delta(\varphi - \varphi_{m}) \), for \( m \)th-order Bessel beams \( A(\varphi) = \exp(i m \theta) \), and for \( m \)th-order Mathieu beams \( A(\varphi; q) = c_{m}(q; q) + i s_{m}(q; q) \), where \( c_{m} \) and \( s_{m} \) are the angular Mathieu functions and \( q \) is the ellipticity parameter.6–8 To get the fundamental PIOF in parabolic coordinate coordinates the task is to find the corresponding spectrum.

Finding PIOFs in parabolic cylinder coordinates requires one to solve the HE in this system. Parabolic cylinder coordinates are defined by the transformation \( x + iy = (\eta + i \xi)^2/2 \) and \( z = z \) with ranges in \( \eta \in (-\infty, \infty) \); \( \xi \in [0, \infty) \); \( z \in (-\infty, \infty) \). In these coordinates the three-dimensional HE separates into a longitudinal part, which has a solution with dependence \( \exp(-ik_z z) \), and a transverse part, which splits into

\[ \frac{d^2 \Phi(\eta)}{d\eta^2} + (k_z^2 \eta^2 + 2k_z a) \Phi(\eta) = 0, \]  

\[ \frac{d^2 R(\xi)}{d\xi^2} + (k_z^2 \xi^2 - 2k_z a) R(\xi) = 0, \]  

where for convenience we have written the separation constant as \( 2k_z a \in (-\infty, \infty) \). By a simple change of variables, \( \sigma \xi \rightarrow v \), where \( \sigma = (2k_z)^{1/2} \), Eqs. (2) and (3) can be transformed into the canonical form of the parabolic cylinder differential equation (PCDE), namely, \( d^2 P/dv^2 + (v^2/4 - a) P = 0 \). The solutions to the PCDE are found by standard methods (e.g., Frobenius), and its Taylor expansion with \( v = 0 \) is
given by\(^1\)

\[
P(v,a) = \sum_{n=0}^{\infty} c_n \frac{v^n}{n!}, \quad c_{n+2} = ac_n - \frac{n(n-1)c_{n-2}}{4}.
\]

We denote the even and odd solutions of the PCDE as \(P_e\) and \(P_o\), respectively. The first two coefficients for the \(P_e\) function are \(c_0 = 1\) and \(c_1 = 0\), whereas for the \(P_o\) solution they are \(c_0 = 0\) and \(c_1 = 1\). For large values of argument \(v\) these solutions have an oscillatory behavior and an envelope that decays as \(v^{-1/2}\).

We emphasize that our approach to expressing the solutions of the PCDE differs from the common methods found in the literature.\(^3,10,12\) In the typical approach the PCDE is transformed into a Weber differential equation whose solutions are the parabolic cylinder functions \(D_i(\mu)\), where \(\nu\) and \(\mu\) are in general complex parameters. Although mathematical connections between the \(P_e\) and \(P_o\) functions with the \(D_i\) functions can be established, we believe that the Weber approach makes the physics confusing and difficult to extract. Instead we prefer to deal with the real and physically achievable \(P_e\) and \(P_o\) solutions.

In searching for solutions in free space, we see that the discontinuity of the \(\eta\) coordinate at the positive \(x\) axis imposes the condition that only products of functions of the same parity in \(\eta\) and \(\xi\) are continuous in the whole space. Then the first (even) and second (odd) transverse stationary solutions are

\[
U_e(\eta, \xi; a) = \frac{1}{\pi^{\nu/2}} |\Gamma_1|^2 P_e(\sigma \xi; a) P_e(\sigma \eta; -a),
\]

\[
U_o(\eta, \xi; a) = \frac{2}{\pi^{\nu/2}} |\Gamma_3|^2 P_o(\sigma \xi; a) P_o(\sigma \eta; -a),
\]

where \(\Gamma_1 = \Gamma[(1/4) + (1/2)ia]\), \(\Gamma_3 = \Gamma[(3/4) + (1/2)ia]\), and the coefficients in front of both equations are for normalization purposes. There are some mathematical properties of the transverse solutions to be discussed here. The solutions are orthonormal with respect to eigenvalue \(a\); this means that \(\int \int U_{e,o}(\eta, \xi; a) U_{e,o}^*(\eta, \xi; a') d\eta d\xi = \delta(a - a')\), where \(\delta\) is the Dirac delta function and the integration is carried out over the whole space. Therefore, analogous to plane waves, Bessel beams, and Mathieu beams, parabolic beams form a complete set of functions in the sense that any PIOF can be represented as a linear superposition of fundamental parabolic beams. Note that parabolic beams, such as plane waves, have a continuous eigenvalue spectrum of multiplicity 2.

Transverse patterns of parabolic beams for \(a = 0, 1.5, -4\) are shown in Fig. 1. The fields exhibit well-defined parabolic nodal lines. Note that the patterns are symmetrical about the \(x\) axis, i.e., \(U_{e,o}(x, -y; a) = \pm U_{e,o}(x, y; a)\), where the plus and minus signs correspond to the even and odd fields, respectively. Evidently the odd solutions vanish along the whole \(x\) axis for any value of \(a\). For the fundamental mode \(a = 0\) the \(y\) axis is also an axis of symmetry. For \(a > 0\) a dark parabolic region around the positive \(x\) axis is present. As \(a\) increases, this region grows, its vertex recedes from the origin, and the field inside it decreases asymptotically to zero. The solutions for \(a < 0\) have a similar behavior, since fields satisfy the symmetry relation \(U_{e,o}(x, y; -a) = U_{e,o}(-x, y; a)\).

After the form of the parabolic PIOFs was obtained, we proceeded to find their spectrum. The approach we used is based on group theory. It can be shown that applying second-order symmetry operators to the solution of the HE and transforming to frequency space allows the spectrum of the parabolic PIOFs to be obtained.\(^4,5\) The spectra for the even and odd solutions are

\[
A_e(\varphi; a) = \frac{1}{2(|\sin \varphi|)^{1/2}} \exp\left(ia \ln |\tan \frac{\varphi}{2}\right),
\]

\[
A_o(\varphi; a) = \frac{1}{i} \begin{cases} 
A_e(\varphi; a), & \varphi \in (-\pi, 0) \\
A_o(\varphi; a), & \varphi \in (0, \pi) 
\end{cases}
\]

An interesting feature of these spectra is that they are expressed in terms of elementary functions only.

![Fig. 1. Transverse patterns of the stationary even and odd parabolic beams for \(a = 0, 1.5, -4\).](image)

![Fig. 2. (a) Amplitude and (b) phase of the angular spectra of even (solid curve) and odd (dashed curves) parabolic beams with \(a = 1\).](image)
The symmetry properties of the amplitude and phase of the angular spectra are shown in Fig. 2. Both amplitude and phase are singular at $\varphi = 0, \pm \pi$ for any value of $a$, with the exception of $a = 0$, for which the phase is zero although the amplitude is still singular. The rapid growth caused by the tangent function at the singular points is slowed down by the log function, resulting in a rather smooth phase.

Until now we have assumed that the fields are of infinite extent; however, in real physical situations the fields have finite transverse extension, introducing diffraction effects into the evolution of the otherwise PIOF. To take this into account, we simulated the propagation of an even parabolic beam, using as the initial condition the field depicted in Fig. 1(b) for $a = 1.5$, through a circular aperture of radius 6 mm. We chose the parameters of the simulation to yield a geometric maximum propagation distance of $\sim 6.0$ m, assuming an illumination at $\lambda = 632.8$ nm and a spatial frequency of $k_l = 10,000$ m$^{-1}$. The evolution along the plane $(x, z)$ is shown in Fig. 3. Note the characteristic cone-shaped region where constituent plane waves superpose to build up the parabolic beam. As part of the family of PIOFs the parabolic beams also reconstruct themselves after they have been partially blocked, as is known to occur with Bessel beams and Mathieu beams.

From the stationary beam solutions described by Eqs. (5) and (6) it is possible to construct traveling solutions of the form $TU^\pm(\eta, \xi, a) = U_0(\eta, \xi, a) \pm iU_1(\eta, \xi, a)$ whose associated spectra are $A^\pm(\varphi, a) = A_0(\varphi, a) \pm iA_1(\varphi, a)$, in which the sign defines the traveling direction. Plots for $a > 0$ are shown in Fig. 4. The traveling-wave feature can be observed in the gradient of their phase structure.

In conclusion, we have shown that parabolic beams complete the family of fundamental PIOFs that are exact solutions of the HE. Analogous to plane waves, Bessel beams, and Mathieu beams, parabolic beams also form an orthonormal and complete set in the sense that any generalized PIOF can be expanded in terms of parabolic beams. We have demonstrated that parabolic beams have a continuous eigenvalue spectrum. Using group theory, we have deduced their angular spectrum in frequency space. We also constructed traveling solutions as a superposition of parabolic beams. Finally, simulations of the propagation of the parabolic beams verified their nondiffracting behavior.

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